Barrier options pricing under stochastic volatility using Monte Carlo simulation

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ABSTRACT

The aim of this paper is to evaluate barrier options by considering volatility as stochastic following the CIR process used in Heston (1993). To solve this problem, we used Monte Carlo simulation. We studied the effects of stochastic volatility on the value of the barrier option by considering different values of the determinants of the option. We illustrated these effects in twelve graphs. We found that in general, regardless of the parameter under study, the stochastic volatility model significantly overvalues the in-the-money (ITM) barrier options, and slightly the deep-in-the-money (DIP) options, while slightly undervaluing the near-out-the-money (NTM) options.

Introduction

Barrier options are very popular and widely used in financial markets. Indeed, the family of barrier options includes a wide variety of options frequently used in practice because they have two major advantages, namely their very attractive prices compared to ordinary options with similar characteristics, as well as their great flexibility to meet investor’s expectations (K Cheng, 2003). A barrier option provides the holder with the right to buy (call option) or sell (put option) a certain amount of an underlying asset at a given price and given maturity initially determined and this conditionally to the hypothetical crossing of a threshold also predetermined (the barrier). It is therefore a so-called path-dependent option, that is to say an option whose gain is based on the trajectory followed by the underlying asset along its life period. The valuation of options and in particular the barrier options, presents an interest both for academicians and practitioners in the financial markets. In this sense, in 1973 Black & Scholes proposed the first method of pricing vanilla options in which the volatility of the underlying asset was considered constant (Black & Scholes, 1973). In the same option pricing context, Merton in 1973 (Merton, 1973) suggested another closed formula that allows the pricing of Call Down and Out barrier options. Subsequently, the evaluation of other types of barrier options, such as activating, deactivating, up and down, was established by Reiner and Rubinstein in 1991 (Rubinstein & Reiner, 1991). The existence of several standard option pricing models has led to the diversification of barrier options assessment procedures such as the binomial tree model proposed by Cox, Ross and Rubinstein in 1979 (M Rubinstein, 2000). However, Boyle in 1986 showed that use of this model did not allow for a rapid convergence of the price of the barrier option (P Boyle, 1986). This led him, therefore, to propose another useful model for the location of the barrier named trinomial tree model. On the other hand, the latter had limits especially when the level of the barrier is close to or far from the initial price of the underlying asset. Subsequently, this failure was noted by Ritchken in 1995 (P Ritchken, 1995). So far, all the aforementioned models are the first-generation models since they were developed under the constraint that volatility is constant in time or so-called deterministic. On the other hand, by studying the financial series we can see that they have several properties that cannot be studied under constant volatility. Thus, the reality of the financial market rejects the assumption of a normal distribution of market returns and invalidates, therefore, the Black-Sholes model and the other valuation models that consider volatility as a deterministic functional and that give bias values. Indeed, this disadvantage becomes very problematic when it comes to evaluating exotic options such as barrier options. To overcome this, several models, namely the models of local volatility or stochastic volatility, were set up and aimed at introducing the specific behavior of volatility. In 1993, Heston proposed a stochastic
volatility model for the valuation of options that is more robust and more in line with the reality of the markets and in particular the consideration of the volatility smile by considering it as stochastic (S Heston, 1993).

In our study we solve the problem of pricing barrier options by considering volatility as stochastic, using the Monte Carlo simulation. We studied the effects of stochastic volatility on the value of the barrier option for various values of the determinants of the option. We illustrated these effects in twelve graphs. The rest of the paper is structured into three sections as follows. The first section highlights the Heston’s model when volatility cannot be considered as a deterministic variable, and which corrects certain failures of the deterministic models. Then we carry out a practical application of this model based on the Monte Carlo simulation for the resolution of differential stochastic equations to present the results obtained in order to compare them with the results obtained when volatility is constant by varying each time one variable in the second section before drawing some conclusions in the third section.

Barrier Option Pricing Model under Stochastic Volatility

Stochastic volatility models provide a better description of the dynamics followed by the price of the underlying asset and can provide a more accurate valuation. The model presented by Heston is a stochastic volatility model characterized by volatility following a CIR random diffusion process. This process considers leverage effect and mean reverting (Tian, Yang, & Zhang, 2019). It assumes that the volatility of the volatility and asset return is arbitrary. Therefore, according to the Heston model (Hi & Lin, 2021), and considering the risk-neutral hypothesis, the price of the financial asset $S_t$ and its variance $v_t$ are assumed to follow the familiar square-root process used by Feller (W Feller, 1951) and Cox, Ingersoll and Ross (Cox, Ingersoll, & SA Ross, 1985):

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{s,t}$$
$$dv_t = \kappa(\theta - v_t) dt + \sigma \sqrt{v_t} dW_{v,t}$$
$$E(dW_{s,t}dW_{v,t}) = \rho dt$$

Where $dW_{s,t}$ and $dW_{v,t}$ are two correlated standard Brownian motions (JP Kahane, 1998). Their correlation is giving by the parameter $\rho$ and it is constant. The three variables $\kappa, \theta, \sigma > 0$ successively represent the mean-reverting speed, the long-run mean and the variation of volatility. The parameter $\mu$ is the instant trend of the stock and is assumed to be constant. Regarding the square-root process, the variance always remains positive if $2\kappa\theta \geq \sigma^2$ and does not cancel out.

We name $t$ the current time, $T$ the maturity time of the option and $\tau = T - t$ the time to maturity. According to Chiarella (Yao & Zhongfeng Qin, 2020), the price of a barrier option when the volatility is stochastic, $C(S, v, t)$, $0 \leq t \leq T$, can be given as a solution of the partial differential equation (PDE) below:

$$\frac{\partial C}{\partial \tau} = \frac{1}{2} v_t S_t^2 \frac{\partial^2 C}{\partial S_t^2} + \rho \sigma v_t S_t \frac{\partial^2 C}{\partial S_t \partial v_t} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 C}{\partial v_t^2} + (r - q) S_t \frac{\partial C}{\partial S_t} + (\kappa(\theta - v_t) - \lambda) \frac{\partial C}{\partial v_t} - rC$$

Where $\lambda$ is the market price of the volatility risk, $r$ is the interest rate and $q$ is the continuous dividend rate. If we name $K$ the strike price, the option has the following terminal condition: $C(S_T, v_T, T) = (S_T - K)^+$. If we name $H$ the barrier level, the domain for the up and out call option is: $0 \leq S_t \leq H; v_t \geq 0; 0 \leq t \leq T$. The barrier option without the early exercise features has the following boundary conditions: $C(S_t, v_t, 0) = (S_T - K)^+$ if $\text{Max}(S_T; 0 \leq t \leq T) < H$. The option contract is defined when the price of the underlying asset reaches the barrier. In this case the owner of the option can exercise it or allow the option be knocked-out (Chiarella & al, 2012), then we estimate that there is no discount and the option will therefore be exercised once the price of the asset reaches the barrier.

If we name $D$ the dividend related to the underlying asset: $D = S(1 - e^{-\sigma \tau})$, when the volatility is constant (i.e. $\kappa = 0$, and $\sigma = 0$), the solution of the PDE is the following formula:

$$\omega = 0.5 + \frac{r - D}{v_0}$$
$$Ln\left(\frac{H^2}{S_0K}\right) = \frac{S_0K}{\sqrt{v_0\sigma}} + \omega \sqrt{v_0\sigma}$$
$$y_1 = \frac{y_1}{\sqrt{v_0\sigma}} + \omega \sqrt{v_0\sigma}$$
$$y_2 = y_1 - \sqrt{v_0\tau}$$

$$X_1 = S_0 e^{-D\tau} \left(\frac{H}{S_T}\right)^{2\omega}N(y_1)$$
$$X_2 = K e^{-r\tau} \left(\frac{H}{S_T}\right)^{2(\alpha-1)}N(y_2)$$
$$C = X_1 - X_2$$

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In case of stochastic volatility, the barrier option price, as a solution of the PDE, can be determined using the Monte Carlo simulation (KS Moon, 2008). If we set \( Y_t = \ln \left( \frac{S_t}{S_0} \right) \), in a risk-neutral world, taking into account the dividend rate \( q \), we have

\[
Y_t = \alpha(t) + \sqrt{1 - \rho^2} \int_0^t \sqrt{\nu_s} \, dw_s \quad \text{with} \quad \alpha(t) = \left( r - q \right) t - \frac{1}{2} \int_0^t \nu_s \, ds + \frac{\rho}{\sigma} \int_0^t \left( v_s - v_0 - \kappa \theta t + \kappa \int_0^t v_s \, ds \right) ,
\]

If we name \( M_s = \max(Y_s; 0 \leq s \leq t) \), \( k = \ln \left( \frac{K}{S_0} \right) \), \( b = \ln \left( \frac{H}{S_0} \right) \), at maturity time \( T \), the pay-off can be written as follows:

\[
\text{Pay-off} = \left( S_0 e^{\nu_T} - K \right) 1_{(M_T < b; Y_T > k)}
\]

and then the option price can be calculated as follows:

\[
C = e^{-rT} E \left[ \xi_v / F_s \right] \text{ where } \xi_v = E \left[ \left( S_0 e^{\nu_T} - K \right) 1_{(M_T < b; Y_T > k)} / F_v \right]
\]

where \( F_s \) and \( F_v \) are the filtrations, respectively, on the underlying asset price \( S_t \) and on the variance \( \nu_t \).

### Results and Interpretations

The assumptions of the constancy of volatility and the log-normal distribution of returns offered by conventional models are very restrictive and poorly reflect the empirical reality. We name \( \Delta \text{C vs} \) the difference between the Barrier option call up and out value \( \text{C vs} \) under stochastic volatility and \( C \) the one under constant volatility: \( \Delta \text{C vs} = \text{C vs} - C \). So in order to show the importance of taking into account stochastic volatility in the evaluation of the barrier options and its effect on it, we will present the curves that plot the differences \( \Delta \text{C vs} \). After developing the simulation algorithm, the value of the buy-and-sell buy-off option was calculated by varying each time an explanatory variable, namely: volatility’s volatility coefficient \( \sigma \), variance \( v_0 \), time to maturity \( \tau \), correlation \( \rho \), the interest rate \( r \) and the barrier level \( H \) to the mean in order to determine their effect on the value of the barrier option. The reference point of this study is defined by the following values of the parameters.

### Table 1: Reference point by the values of the parameters

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>( d )</th>
<th>( r )</th>
<th>( \sigma )</th>
<th>( H )</th>
<th>( K )</th>
<th>( \tau )</th>
<th>( \theta_v )</th>
<th>( \nu_0 )</th>
<th>( \lambda_v )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.05</td>
<td>0.03</td>
<td>0.1</td>
<td>130</td>
<td>100</td>
<td>0.5</td>
<td>2</td>
<td>0.1</td>
<td>0.1</td>
<td>0</td>
</tr>
</tbody>
</table>

The parity of the option can be divided into four intervals: deep in the money (DIM) near in the money (NIM), near out of the money (NOM) and deep out of the money (DOM).

### The correlation effects

We notice, in (Fig. 1) and (Fig. 2), that there are not big differences between the values of this barrier option for the different correlation levels \( \rho \) and the curves always keep the same pace. This difference is slightly sensitive to the variation of the correlation. Whatever the level of correlation, the stochastic volatility model significantly overvalues the ITM barrier options, while slightly overvaluing them for the DOM options. On the other hand, it slightly undervalues the NOM options. This effect, which reflects the asymmetry of the underlying asset return distribution, decreases with the level of correlation.
The volatility’s volatility coefficient effect

The fluctuation in volatility has a significant effect on the value of the barrier option (Fig 3. and Fig 4.). An increase in the coefficient $\sigma$ that controls the volatility of volatility increases the flattening coefficient of yields, which is Kurtosis. In comparing with the barrier option price under constant volatility, we find that, for the two extreme levels of volatility’s volatility, i.e. $\sigma = 0.1$ and $\sigma = 1$, the difference $\Delta C_{vs}$ evolves inversely.

These results show the significant effect of taking into account the change in volatility in the valuation of barrier options. This is considered as the effect of kurtosis. For volatility’s volatility ($\sigma$) levels of 10% and 50%, the stochastic volatility model significantly overvalues the ITM barrier options, while slightly overvaluing them for DOM options. On the other hand, it slightly undervalues the NOM options. These effects decrease with the volatility’s volatility up to a level $\sigma^*$, to be determined. From $\sigma^*$ the effect of $\sigma$ on the value of the barrier option is reversed. This is the case of the level $\sigma = 100\%$. The shape of the $\Delta C_{vs}$ curve changes with the values of $\sigma$. 

**Figure 1:** Barrier call up and out price depending on the price of the underlying asset for various values of the correlation $\rho$. $v=0.1$, $K=100$, $H=130$, $r=0.03$, $d=0.05$, $\tau=0.5$, $v=0.1$, $\sigma=0.1$, $\lambda v=0$.

**Figure 2:** The difference $\Delta C_{vs}$, depending on the Underlying asset price for various values of the correlation $\rho$. $v=0.1$, $K=100$, $H=130$, $r=0.03$, $d=0.05$, $\tau=0.5$, $Kv=2$, $tv=0.1$, $\sigma=0.1$, $\lambda v=0$.

**Figure 3:** Barrier call up and out price depending on the price of the underlying asset for various values of the volatility’s volatility coefficient $\sigma$ with parameters values: $K=100$, $H=130$, $\tau=0.5$, $r=0.03$, $d=0.05$, $Kv=2$, $tv=0.1$, $\lambda v=0$, $\rho=0.5$.

**Figure 4:** The difference $\Delta C_{vs}$, depending on the underlying asset price for different values of the volatility’s volatility coefficient $\sigma$ with following parameters values: $K=100$, $H=130$, $\tau=0.5$, $r=0.03$, $d=0.05$, $Kv=2$, $tv=0.1$, $\lambda v=0$, $\rho=0.5$.
The variance effect on the barrier option price

As illustrated in (Fig. 5), the barrier call up and out is very sensitive to the variance variations.

![Figure 5](image1.png)

**Figure 5:** Barrier call up and out price depending on the price of the underlying asset for various values of the variance $\nu$ with parameters values: $K=100$, $H=130$, $\tau=0.5$, $r=0.03$, $d=0.05$, $Kv=2$, $\theta v=0.1$, $\sigma=0.1$, $\lambda=0$, $\rho=-0.5$.

In comparing with the barrier option price under constant volatility (Fig. 6), we find that, the decline in variance has the effect of increasing the difference between the prices of the barrier out of the money options and lowering the difference between the barrier at the money and the money option prices. Compared to the constant volatility model, for variance levels below 0.25, the stochastic volatility model overvalues the OTM barrier options, while it undervalues the ITM options. These effects decrease with variance level inferior to a level $\nu^*$, to be determined. From $\nu^*$ the volatility model overvalues all the barrier options whatever their parities, and the shape of the curve $\Delta C_{\nu}$ changes. This is the case of level $\nu = 0.25$.

The time to maturity effect

In (Fig. 7) and (Fig. 8), we notice that the value of the buy-and-sell buy-out option is sensitive to changes in the value of the time to maturity. Indeed, these two values evolve inversely.

![Figure 7](image2.png)

**Figure 7:** Barrier call up and out price depending on the price of the underlying asset for various values of the time to maturity $\tau$. $\nu=0.1$, $K=100$, $H=130$, $d=0.05$, $r=0.03$, $Kv=2$, $\theta v=0.1$, $\sigma=0.1$, $\lambda=0$, $\rho=-0.5$.

![Figure 8](image3.png)

**Figure 8:** The difference $\Delta C_{\tau}$, depending on the underlying asset price for various values of the time to maturity $\tau$. $\nu=0.1$, $K=100$, $H=130$, $d=0.05$, $r=0.03$, $Kv=2$, $\theta v=0.1$, $\sigma=0.1$, $\lambda=0$, $\rho=-0.5$.

The shorter the term of maturity, the higher the value of the barrier option. The shorter the residual life of the option, the greater the difference, and it varies in both directions. For options near the maturity, the stochastic volatility model significantly overvalues the
ITM barrier options, while it overvalues them for the DOM options. From a certain value \( \tau^* \), the shape of the curve \( \Delta C_{vs} \) changes and whatever the parity of the barrier option, the stochastic volatility model overvalues the option comparatively to the constant volatility model.

**Interest rate effect**

As illustrated in (Fig. 9) and (Fig. 10), this difference is slightly sensitive to the change in the interest rate.

**Figure 9:** Barrier call up and out price depending on the price of the underlying asset for various values of the interest rate \( r \). \( v=0.1, K=100, H=130, d=0.05, \tau=0.5, \theta v=0.1, \sigma=0.1, \lambda v=0, \rho=-0.5 \).

**Figure 10:** The difference \( \Delta C_{vs} \), depending on the underlying asset price for various values of the interest rate \( r \). \( v=0.1, K=100, H=130, d=0.05, \tau=0.5, \theta v=0.1, \sigma=0.1, \lambda v=0, \rho=-0.5 \).

Whatever the level of interest rate, the stochastic volatility model significantly overvalues the ITM barrier options, while slightly overvaluing them for the DOM options. On the other hand, it slightly undervalues the NOM options.

**The barrier level effect**

We notice, in (Fig. 11) and (Fig. 12), that for a level of barrier inferior to \( H^* \), to be determined, whatever the level of the maturity, the stochastic volatility model significantly overvalues the barrier options. For a level of barrier superior to \( H^* \), the stochastic volatility model significantly overvalues the ITM barrier options, while slightly overvaluing them for the DOM options. On the other hand, it slightly undervalues the NOM options. For a given option parity, the overvaluation of the option, increases with the Barrier level.

**Figure 11:** The difference \( \Delta C_{vs} \) as a function of the underlying asset price for various values of the barrier level \( H \). \( v=0.1, K=100, H=130, d=0.05, \tau=0.5, \theta v=0.1, \sigma=0.1, \lambda v=0, \rho=-0.5 \).

**Figure 12:** The difference \( \Delta C_{vs} \), depending on the barrier level \( H \), for various values of the underlying asset price \( S \). \( v=0.1, K=100, H=130, d=0.05, \tau=0.5, \theta v=0.1, \sigma=0.1, \lambda v=0, \rho=-0.5 \).
However, for a maturity superior to 1.2, we notice that from a certain barrier level, the stochastic volatility model undervalues the option.

Conclusion

In this paper we have studied the effects of stochastic volatility on the price of a barrier option, initially varying the correlation coefficient between the volatility and the return on the underlying asset that reflects the asymmetry of the aforementioned yield distribution (skewness). In a second step, we studied the effects of the volatility of volatility that reflects the flattening effect of the aforementioned distribution (Kurtosis). Thirdly, we studied the effects of the variance, the interest rate or the time to maturity and the level of the barrier. We found that in general, regardless of the parameter under study, the stochastic volatility model significantly overvalues the ITM barrier options, and slightly the DOM options, while slightly undervaluing the NOM options. For certain parameters, such as the volatility of volatility, the interest rate or the time to maturity, the effect of stochastic volatility on price reverses beyond a certain level of one of these parameters. Thus, we have underlined the importance of using the Heston model, which considers the random nature of volatility, and which offers a better evaluation of the barrier options compared to deterministic models. Finally, the extension of our future research will focus on hedging strategies and financial risk management based on these barrier options.

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Data Availability Statement: The data presented in this study are available on request from the corresponding author. The data are not publicly available due to privacy.

Conflicts of Interest: The authors declare no conflict of interest.

References


Appendix

Table A: Variable list

<table>
<thead>
<tr>
<th>Variable</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(t)</td>
<td>asset price at time t</td>
</tr>
<tr>
<td>D</td>
<td>the dividend</td>
</tr>
<tr>
<td>r</td>
<td>risk free interest rate</td>
</tr>
<tr>
<td>\sqrt{v(t)}</td>
<td>volatility of the asset price</td>
</tr>
<tr>
<td>\sigma</td>
<td>volatility of the \sqrt{v(t)}</td>
</tr>
<tr>
<td>H</td>
<td>the barrier level</td>
</tr>
<tr>
<td>\tau</td>
<td>the time to maturity</td>
</tr>
<tr>
<td>T</td>
<td>the maturity</td>
</tr>
<tr>
<td>\rho</td>
<td>the correlation</td>
</tr>
<tr>
<td>\lambda</td>
<td>the price of volatility risk</td>
</tr>
<tr>
<td>\theta</td>
<td>the long run variance of volatility</td>
</tr>
<tr>
<td>k</td>
<td>the mean reversion of volatility</td>
</tr>
<tr>
<td>K</td>
<td>the strike price</td>
</tr>
</tbody>
</table>

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